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Optimization of friction surfaces through mathematical modelling of the flow of lubricants

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KEYWORDS

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ABSTRACT

The aim of the research was to determine the optimal friction parameters in the tribological pair “friction surface + flowing lubricant”. The research employed a methodological approach centred on the application of mathematical tools, encompassing mathematical modelling and numerical computations. The comparison of numerical tests on the coefficient of friction for the typical and optimized design of the sliding bearing yields the following conclusions: Both sets of studies demonstrate a comparable pattern in the behavior of the function, with a distinct ideal range between 4 and 8 MPa of hydrodynamic pressure. By integrating a suitable coating made of a low-melting alloy, the constructive optimization of the sliding bearing resulted in an average reduction of the friction coefficient by 22.1% and a reduction of 90.1% in the optimal range. The interpretation of the results highlights the significant influence of technological properties of the frictional surface on the friction coefficient.

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1.0 INTRODUCTION

The production of lubricants over the past 10 years has had a steady upward trend (Figure 1) due to increasing consumer demand (https://www.statista.com) from 35.3 million tons in 2013 to 37.6 million tons in 2021.

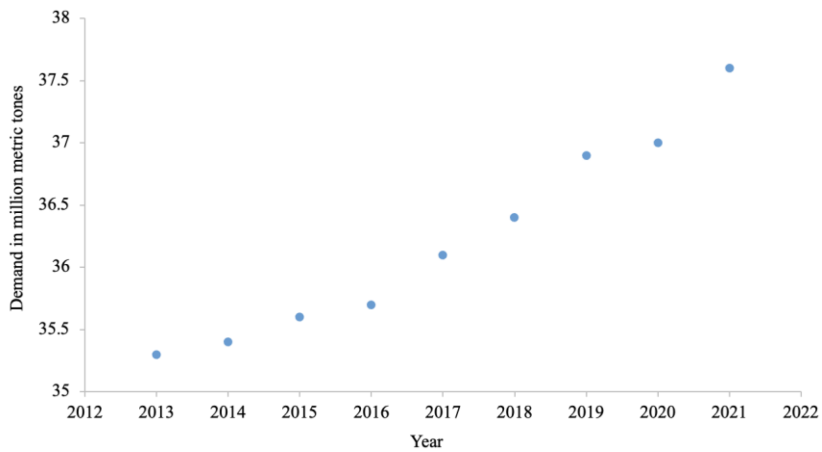

Figure 1: The production of lubricants over the past 10 years.

The investigation of friction surface optimization through mathematical modelling of the flow behaviour of flowing lubricants under variable pressures is pertinent and significant for several reasons:

Enhancement of lubricant efficiency: Lubricants play a crucial role in mitigating friction and wear in various mechanisms and machines. Optimizing friction surfaces can lead to improved lubricant efficiency, enabling them to exhibit lower friction and wear levels under variable pressure conditions. This, in turn, would extend the service life and reliability of mechanisms while reducing energy consumption (Alahmer and Ghoniem, 2023; Verma et al., 2023; Benzaid et al., 2023).

Reduction in operational costs: Improving friction surfaces can reduce operational costs across various industrial sectors. By diminishing friction and wear, the need for frequent component replacements and repairs can be minimized, thereby reducing expenses on spare parts and maintenance. This holds particular significance for large-scale industrial systems, encompassing aviation, the automotive industry, energy, and manufacturing (Marlinda et al., 2023; Sukumaran et al., 2023; Fakhri et al., 2023).

Ecological significance: The enhancement of lubricant efficiency can also yield positive environmental impacts. Reducing friction and wear leads to decreased energy consumption, thereby contributing to the reduction of greenhouse gas emissions and other pollutants. Additionally, increased lubricant efficiency may decrease the volume of waste generated from their usage and replacement (Virdi et al., 2023; Negi et al., 2023; Okubo et al., 2023).

Technological progress: Research endeavours related to friction surface optimization foster technological progress in the field of lubricants and tribology as a whole. The development of novel mathematical models and analytical methods allows for a more precise and efficient examination of the interaction between lubricants and friction surfaces. This can lead to the creation of new lubricants, advancements in designs, and optimization of production processes (Adetunla et al., 2023; Seman and Gözen, 2023; Djas et al., 2022).

Thus, the investigation of friction surface optimization through mathematical modelling of the flow behaviour of flowing lubricants under variable pressures holds significant relevance and potential for application across diverse industrial sectors. This research can enhance lubricant efficiency, reduce operational costs, improve environmental sustainability, and foster technological progress.

2.0 LITERATURE REVIEW

The author of the study (Voitov, 2023) has developed a mathematical model for the frictional heating of tribosystems under boundary lubrication conditions. The identification of the tribosystem's structure and parameters revealed that the heating processes are described by a second-order equation, accounting for stability limits and the tribosystem's stability margin. The amplification coefficients and time constants influence the heating conditions, while the input excitation varies over time and exhibits both positive and negative accelerations concerning the specified values.

The research conducted by Gavrilov et al. (2023) introduces a mathematical modelling method for hydrodynamic friction in plunger pairs, considering the micro geometry of friction surfaces. By employing a modified Reynolds equation and accounting for the cavitation effect of the lubricating fluid, the developed software enables the analysis of contacting plunger surfaces and the assessment of their tribotechnical characteristics based on microgeometry parameters. The study also investigates the influence of micro geometry on the quality of hydro-mechanical properties in plunger pairs and presents computational examples of analyzing contacting plunger surfaces with varying roughness microgeometry parameters.

In the study conducted by Yu et al. (2023), a comprehensive analytical model for the friction coefficient between fractal surfaces under mixed lubrication conditions is presented. A mathematical transformation from the area of asperities to the thickness of the oil film was performed, altering the traditional foundation of the fractal method based on area distribution and establishing an analytical fractal model for the oil film thickness. Subsequently, a dynamic film thickness model considering external loading in determining surface roughness was proposed. Leveraging the fractal model of oil film thickness, the contact force was solved for each portion under the influence of external normal force, leading to a decomposition represented by the contact relation. The constitutive model for the friction coefficient was determined via the contact relation. The effects of fractal parameters and critical operating conditions on the friction coefficient were analyzed, and comparisons with experimental data were conducted.

The research conducted by Shahzad et al. (2023) presents a mathematical model of nanofluid based on graphene oxide/kerosene, utilizing a linear expansion radiation surface. The study encompasses applications of a parabolic trapezoidal heat storage collector (PTSC) in various systems. Employing the Newtonian nanofluid model, the researchers optimize the operational parameter of heat transfer in the aerospace industry. Experimental investigations with graphene oxide and kerosene demonstrate that such nanofluid exhibits higher efficiency compared to conventional fluids.

The research conducted by Hu et al. (2023a) focuses on friction modelling in cold forging under various tribological conditions. The study confirms a generalized friction model through a combination of experimental and numerical investigations on semi-conical specimens. The investigation reveals that different friction models lead to varying stress-dependent friction distributions. Additionally, mathematical models for friction were developed, considering normal pressure and contact temperature, as well as a model accounting for sliding between the tools and the workpiece. Leveraging data acquired from different tribometers, distinct friction models were established to correlate tribological loads with the friction coefficient.

In the study by Engao et al. (2023), lubrication characteristics of friction between the piston ring and the cylinder liner during piston motion are analyzed. Utilizing a hydrodynamic model that considers variations in oil viscosity and density, parameters describing lubrication characteristics during piston motion were derived. The investigation revealed that improving the surface roughness of the piston ring enhances lubrication while increasing oil viscosity aids in improving lubrication characteristics and reducing peak friction. Furthermore, it was found that enhancing the surface texture of piston rings can reduce the oil film thickness and mitigate friction in regions of high-speed motion.

The research conducted by Liang et al. (2023a) presents a mixed lubrication model for an elastically supported water-lubricated tilting thrust bearing. The model accounts for the elastic-plastic deformation of asperities, the polymer matrix of the bearing pad, and the elastic deformation of the rubber support. The study demonstrates that surface roughness influences the contact area and lubrication state but has little effect on the average film thickness. The mixed lubrication model predicts the film thickness and other lubrication characteristics parameters closer to the measured results compared to other models.

The research conducted by Mukutadze and Lagunova (2022) introduces a mathematical model of lubrication in a bearing with a fusible coating on the guide and an uneven profile of the sliding element. An asymptotic solution to the system of differential equations is developed, considering the melting of the coating and the profile of the guide. Novel expressions for bearing characteristics are proposed, incorporating the melting of the coating and incomplete filling of the working clearance. The obtained models allow for the regulation of load-carrying capacity and friction coefficient by varying the low-melting coating on the guide's surface.

The study by Cai et al. (2022) developed a mathematical model to investigate nonlinear dynamic mixed friction in a novel connected bearing lubricated with a low-viscosity fluid. The model incorporates hydrodynamic forces, contact forces, deformation, and displacements. Numerical simulations were conducted to reveal dynamic nonlinear phenomena of mixed friction in such a bearing and to explore the influence of structural parameters on these phenomena. The results demonstrated that exciting moments and angular displacements significantly influence the characteristics of mixed friction. Increasing the length-to-diameter ratio of the bearing enhances the nonlinear dynamic properties of mixed friction, whereas increasing the radius of the supporting bearing improves friction characteristics in the journal bearing. In Table 1, we collected mathematical models developed by different authors.

Currently, the modelling of the tribological pair “friction surface + flowing lubricant” under the influence of varying system pressure encounter challenges due to the lack of an adequate and well-founded mathematical model, whose mathematical framework would enable a closer approximation to the real conditions of modelling the investigated object.

Given this circumstance, the justified objective of this research is to develop a mathematical framework for model-based optimization of the tribological pair “friction surface + flowing lubricant” under the influence of varying system pressure.

The research objectives are as follows: development of fundamental mathematical formulations establishing the basis for the designed mathematical framework; creation of a mathematical model for the investigated tribological pair; Optimization modelling and validation of the developed mathematical framework; Comparative analysis of model-based and experimental data.

Table 1: Mathematical models.

Reference	Describing of mathematical model
Voitov (2021)	Frictional heating of tribosystems under boundary lubrication conditions
Gavrilov et al. (2023)	Hydrodynamic friction in plunger pairs, considering the microgeometry of friction surfaces
Yu et al. (2023)	Friction coefficient between fractal surfaces under mixed lubrication conditions
Shahzad et al. (2023)	Nanofluid based on graphene oxide/kerosene, utilizing a linear expansion radiation surface
Hu et al. (2023)	Cold forging under various tribological conditions
Engao et al. (2023)	Hydrodynamic model that considers variations in oil viscosity and density
Liang et al. (2023)	Elastically supported water-lubricated tilting thrust bearing
Mukutadze et al. (2022)	Lubrication in a bearing with a fusible coating
Cai et al. (2022)	Nonlinear dynamic mixed friction

3.0 METHODS AND MATERIALS

The mathematical modelling of the tribological pair “friction surface + flowing lubricant” is based on equations that describe the physical phenomena occurring within this system. The model takes into consideration various factors, such as hydrodynamic pressure, boundary conditions, elastic surface deformations, rheological properties of the lubricant, and others. The development of the mathematical framework is presented in supplementary part of the article.

Frictional force:

$$F_{mp} = \mu_0 \times \left(1 - a \times p + \frac{a^2 \times p^2}{2}\right) \times \int_{\theta_1}^{\theta_2} \left(\left.\frac{\partial v_0}{\partial r}\right|_{r=0} + K \times \left.\frac{\partial v_1}{\partial r}\right|_{r=0} \right) d\theta.$$

Thus, a mathematical model describing the contact zone of the investigated tribological pair “frictional surface + flowing lubricant” has been obtained, and based on the results, it is possible to obtain optimization solutions for the frictional surfaces.

As a result of mathematical transformations (see supplementary part), a mathematical model has been derived for optimizing frictional surfaces through the mathematical modelling of the flow of flowing lubricants under variable pressures.

The mathematical model describing the flow of flowing lubricants under variable pressures can be utilized for optimizing frictional surfaces in the following ways:

1. Surface Geometry Optimization: The model can be employed to assess the influence of surface geometry on pressure distribution and flow velocity of the lubricant material. This allows investigating various shapes and parameters of frictional surfaces and optimizing them to achieve desired tribological characteristics, such as minimizing friction and wear.
2. Lubricant Material Properties Optimization: The model can be used to study the impact of lubricant material properties, such as viscosity, density, and temperature, on friction and wear. This enables the optimization of lubricant material composition and parameters to attain optimal tribological outcomes.
3. Operation Conditions Optimization: The model can be utilized to determine optimal operating conditions, such as pressure, velocity, and temperature, which ensure optimal lubrication and minimal friction in the system. By employing the model, one can investigate the influence of varying these parameters on friction and wear characteristics and select the optimal conditions for a specific application.
4. Materials and Coatings Optimization: The model can be employed for comparative analysis of different materials and coatings and determining their impact on friction and wear. This facilitates the selection of the most suitable materials and coatings to enhance the efficiency of friction systems.

All these aspects of modelling contribute to the optimization of friction surfaces, ensuring more efficient lubrication, reduced friction and wear, as well as increased reliability and durability of friction systems. For the validation of the formulated mathematical model, appropriate numerical investigations will be conducted for the fourth aspect of potential variations in mathematical modelling – aimed at establishing optimization solutions for materials and coatings of friction surfaces.

As the subject of numerical investigation, we consider a typical configuration of a friction surface (sliding bearing) operating in contact with a flowing lubricant while varying the pressure in the modelled tribological pair. The numerical validation of the formulated mathematical model was performed using digital tools for mathematical computation and simulation – PC MathCad.

The objective of the numerical investigation (during the validation stage of the experimentally theoretical model) is to obtain solutions for optimizing the friction surface (within the contact zone of the previously mentioned tribological pair) by using optimal materials and exploring the rheological properties of the lubricant and assessing their influence on the modelled system. The numerical study is conducted while varying the hydrodynamic pressure in the system. The evaluation of the effectiveness of the optimization solutions is carried out based on the parameter of the friction coefficient.

4.0 RESULTS

Initial results were obtained for a typical structural solution of a plain bearing - a sleeve with a metallic coating - as shown in Figure 2.

$$y = -4 \times 10^{-6}x^4 + 9 \times 10^{-5}x^3 - 0.0008x^2 + 0.023x$$

$$R^2 = 0.9779$$

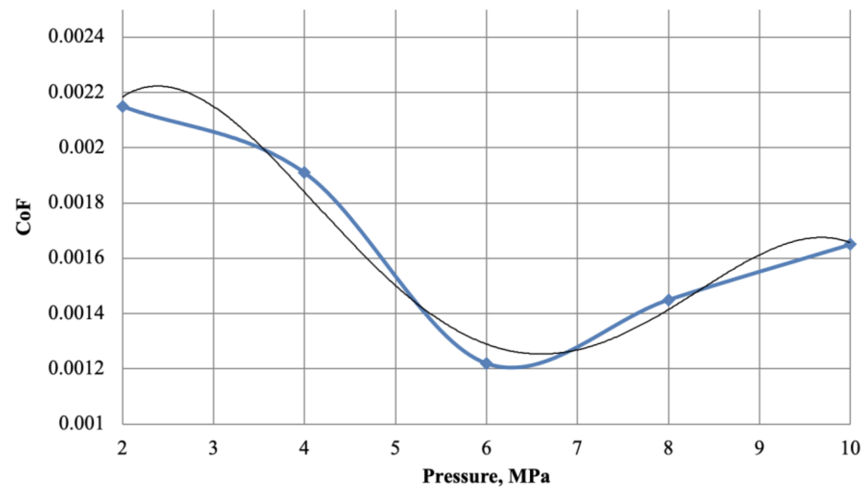

Figure 2: Results of the first series of numerical experiments for a typical structural solution of a plain bearing (sleeve with a metallic coating).

As an optimization solution, we apply the modernization of the typical structural design of the simulated plain bearing by integrating a layer of low-melting alloy – Wood's alloy - onto the sleeve.

Wood's alloy is a low-melting alloy composed of approximately 50% tin (Sn), 25% lead (Pb), 12.5% bismuth (Bi), and 12.5% cadmium (Cd). It exhibits a low melting temperature, approximately 60-70°C, and a density of about 9.33 g/cm^3. Wood's alloy finds extensive applications in various fields, including metal casting, electronics, medicine, and others, owing to its low melting temperature and ease of processing. Metal alloys such as Wood's metal have lower melting points than their components due to the presence of other elements in the alloy. These elements can disrupt the crystal structure of the metal, weakening the metallic bonds between atoms and making the alloy more susceptible to melting at a lower temperature. Additionally, the presence of other elements in the alloy can also affect the atomic packing density and the coordination number of the atoms, also contributing to a lower melting point. It also forms good joints with other materials and can be used as a soldering material for joining metal parts (Huang et al., 2019; Singh et al., 2018; Handschuh-Wang et al., 2023).

Results of numerical modelling for the optimized structural solution:

$$y = -2 \times 10^{-5}x^4 + 0.0005x^3 - 0.0044x^2 + 0.0142x - 0.0128$$

$$R^2 = 1$$

The comparative analysis of numerical experiments on the investigated coefficient of friction for the typical and optimized design of the sliding bearing (Figure 4) leads to the following conclusions:

Both series of experiments exhibit a similar characteristic trend of the function, with a characteristic optimal range between 4 and 8 MPa of hydrodynamic pressure: the operation of the modelled bearing outside this range leads to an increase in the friction coefficient and a reduction in the effectiveness of the sliding mechanism (friction);

The function $\mu = f(p)$ for both series of numerical experiments is well-described by a polynomial dependence (4th degree) with a high level of reliability (based on the coefficient of determination);

The technological properties of the friction surface have a more significant impact on the contact zone parameters (friction coefficient) compared to the rheological properties of the lubricant.

The constructive optimization of the sliding bearing by integrating an appropriate coating made of low-melting alloy resulted in an average reduction of the friction coefficient by 22.1% and a reduction of 90.1% in the optimal range.

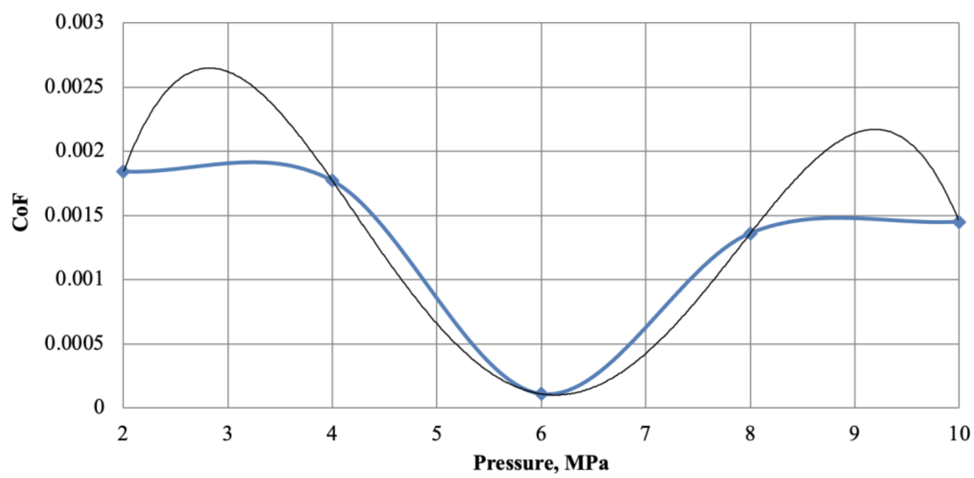

Figure 3: Results of the second series of numerical experiments for the optimized design solution of the sliding bearing (bushing with a coating of low-melting alloy).

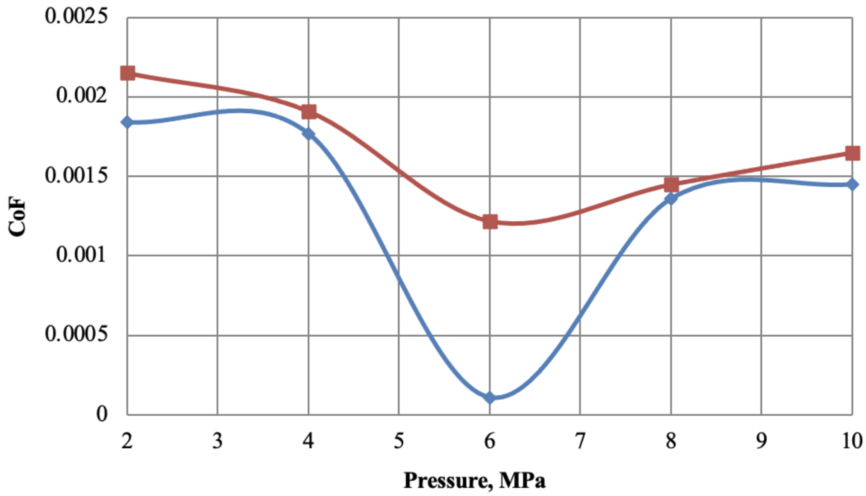

Figure 4: Comparative analysis of the results from two series of numerical experiments for the conventional and optimized design of the sliding bearing.

Thus, the results of numerical experiments for the developed mathematical model enable obtaining optimization solutions not only for the frictional surface but also potentially for other components of the investigated tribological system. The developed mathematical model for the optimization of frictional surfaces by mathematical modelling of fluid flow in lubricating materials under varying pressures has potential applications and opportunities:

Prediction and optimization of friction: The model allows predicting and optimizing the coefficient of friction based on various factors such as pressure, velocity, and lubricating material properties. This enables researchers and engineers to develop more efficient lubrication systems and frictional surfaces.

Improving lubrication efficiency: The model can be used to determine optimal lubrication parameters, such as pressure and velocity, to achieve the best lubrication efficiency. This helps reduce wear and enhance the durability of the contacting surfaces.

Optimization of frictional surface design: The model can aid in optimizing the geometry and microstructure of frictional surfaces to achieve optimal tribological properties. This involves selecting optimal parameters for surface texturing, such as roughness and micro-relief shape.

Investigation of new materials and lubricants: The model can be used to explore the influence of different materials and lubricating substances on friction and wear behaviour. This facilitates the selection of the most suitable materials for specific tribological applications.

Determination of lubrication conditions: The model can assist in determining the optimal lubrication conditions, such as pressure and velocity, to prevent sliding situations and enhance hydrodynamic lubrication. This is particularly valuable for developing low-friction systems and increasing overall efficiency.

The developed mathematical model offers a wide range of opportunities for research and optimization of frictional surfaces and lubrication systems. The model enhances the understanding of friction and lubrication processes, facilitates the development of novel materials and designs, and enables the optimization of frictional surfaces to enhance efficiency and durability in various tribological systems.

5.0 DISCUSSION

A comparative analysis will be conducted between the obtained results and the findings from related studies conducted within the established research direction. Voitov's research (2023) concentrates on developing a thermodynamic model for tribological systems, while our study is focused on optimizing frictional surfaces using mathematical modelling of hydrodynamic lubricant flow. However, both studies share a common goal of seeking optimization solutions for tribological systems through suitable mathematical modelling.

The findings of the research conducted by Gavrilov et al. (2023) are limited in scope as they are applicable for modelling hydrodynamic friction of plunger pairs, considering the micro geometry of frictional surfaces. In contrast, the results of the present study characterize the tribological system of "frictional surface + fluid lubricant", which is more typical for rotating surfaces. However, the compared results aim to obtain essential mathematical relationships that enable the derivation of optimization solutions for various contact systems of frictional surfaces with technological and design considerations.

The focus of the research by Yu et al. (2023) is directed towards modelling the technological parameters of lubricant, taking into account the dynamic thickness of the oil film, external load, and surface roughness of the frictional surface. Consequently, the results obtained by Yu et al. (2023) lead to optimization solutions for the lubricant, whereas the present study aims at the optimization of the frictional surface. However, both studies utilize the coefficient of friction as the primary parameter to assess the efficiency of tribological systems, suggesting a potential complementarity between the compared findings.

The results of the research by Shahzad et al. (2023) are focused on developing a mathematical model for the thermodynamic and rheological properties of lubricants at the nanoscale. In contrast, the results of this study are focused on the properties of the frictional surface, indicating that the obtained technological and design properties of the optimized frictional surface have a more significant impact on the coefficient of friction parameter than the rheological properties of the flowing lubricant in the modelled tribological system.

The research focus of Hu et al. (2023a) is on tribological parameters of cold forging, which cannot be directly compared with the results of the present study due to their different research directions. However, both this study and Hu et al. (2023a) share the common research objective of reducing the influence of frictional forces.

The research by Engao et al. (2023) focuses on modelling the rheological properties of lubricant between the piston ring and the cylinder liner during piston movement, with limited scope considering the specificity of the mentioned tribological system. However, both the study by Engao et al. (2023) and the present research involve modelling the hydrodynamic flow of the flowing lubricant, aimed at reducing the adverse effects caused by frictional forces.

The study by Liang et al. (2023a) is also focused on modelling the rheological properties of lubricants, similar to Engao et al. (2023). However, in this case, the tribological model involves a tilting thrust-bearing system, making the obtained results adaptable in the context of the present research. Both Liang et al. (2023b) and the current study share the assumption that the parameters of the friction surface (including its optimization variations) hold greater significance than the rheological properties of the lubricant.

Furthermore, the main findings of the current study are supported by the results obtained in the research by Mukutadze and Lagunova (2022). However, the focus of the compared study is directed towards the mathematical modelling of the low-melting coating melting process on the bearing bush and the optimization of the technological parameters of the simulated tribological system. Although Mukutadze and Lagunova (2022) exhibit distinct research aspects, their findings may complement the results of the current study regarding the optimization of friction surfaces.

Cai et al. (2022) also investigates the hydrodynamic processes during lubricant flow in a bearing, with a focus on modelling phenomena of nonlinear dynamic mixed friction, taking into account hydrodynamic force, contact force, deformation, and displacements. In contrast, the present study is oriented towards the mathematical description of the friction surface, operating under varying hydrodynamic lubrication pressures. Despite the different aspects of the obtained results, the compared studies are complementary in the context of establishing optimal hydrodynamic flow regimes of lubricant in the simulated tribological system.

Analyzing the results of works (Voitov, 2021; Gavrilov et al., 2023; Yu et al., 2023; Shahzad et al., 2023; Hu et al., 2023; Engao et al., 2023; Liang et al., 2023; Mukutadze et al., 2022; Cai et al., 2022), it can be stated that in works (Voitov, 2021; Yu et al., 2023; Hu et al., 2023; Engao et al., 2023; Liang et al., 2023; Mukutadze et al., 2022; Cai et al., 2022) there are no practical recommendations for their implementation. Work by Shahzad et al. (2023) is also theoretical in nature; it was determined that graphene oxide (GO) based kerosene oil has a heat transition rate of 15.03% higher than conventional fluid without GO. Only in Gavrilov et al. (2023) there are practical research results, namely: friction losses were reduced by 7.5% and the load carrying capacity of the lubricating film was increased by raising the pressure in the lubricating film by 20%. Taking into account the above, we can state that the results we obtained differ significantly from those given in Voitov (2021), Gavrilov et al. (2023), Yu et al. (2023), Shahzad et al. (2023), Hu et al. (2023), Engao et al. (2023), Liang et al. (2023), Mukutadze et al. (2022), Cai et al. (2022) and have practical value. In particular, the R2 values of the developed mathematical models are equal to 1. The developed mathematical model offers a wide range of opportunities for research and optimization of frictional surfaces and lubrication systems. The model enhances the understanding of friction and lubrication processes, facilitates the development of novel materials and designs, and enables the optimization of frictional surfaces to enhance efficiency and durability in various tribological systems.

Thus, the comparative analysis of relevant and current publications allows not only to compare the results of the present study with similar ones (which have led to comparable parameters and mathematical expectations) but also to complement and expand the developed model of the hydrodynamic flow of the lubricant under variable pressure with important aspects concerning optimal hydrodynamic regimes of the simulated tribological system.

CONCLUSIONS

The present study has developed a mathematical model describing the contact zone in the tribological pair “frictional surface + flowing lubricant”. This model enables the derivation of optimization solutions for frictional surfaces through mathematical modelling of lubricant flow under variable pressures. Comparative analysis of the investigated friction coefficient characteristic revealed that the operation of the bearing outside the specified range of hydrodynamic pressures (4 – 8 MPa) leads to an increase in friction and a decrease in the efficiency of the sliding mechanism. The function describing the results of numerical experiments exhibits a polynomial dependence with a high level of reliability. Technological properties of the frictional surface have a stronger influence on the friction coefficient than the rheological properties of the lubricant.

Constructive optimization of the sliding bearing by incorporating a coating of low-melting alloy reduces the friction coefficient by 22.1% on average and by 90.1% within the optimal zone of hydrodynamic pressures in the simulated system. Thus, the results of numerical experiments obtained using the developed mathematical model offer the possibility of obtaining optimization solutions not only for the frictional surface but also for other components of the tribological system.

The practical application of the results lies in the optimization of tribological parameters of frictional surfaces in typical mechanical systems, employing the model of "frictional surface + flowing lubricant". The optimization of technological and design solutions allows for enhanced efficiency of typical mechanisms with contact zones.

The subsequent steps in this research direction involve conducting real-world experiments to obtain empirical relationships and assessing the adequacy of the mathematical model against field data. Additionally, further development of the corresponding mathematical framework is essential to ensure a comprehensive representation of the real physical phenomena in the simulated tribological system. This includes integrating processes of hydrodynamic mixing and accounting for thermodynamic parameters. Furthermore, exploring other identified possibilities for optimizing frictional surfaces, such as geometric optimization, lubricant material property optimization, and operational condition optimization, is of utmost importance.

These endeavours will advance the understanding and application of the mathematical model and contribute to the practical implementation of optimization strategies in various tribological systems.

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